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Dimensional reduction via noncommutative spacetime: bootstrap and holography

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Abstract: Unlike noncommutative space, when space and time are noncommutative, it seems necessary to modify the usual scheme of quantum mechanics. We propose in this paper a simple generalization of the time evolution equation in quantum mechanics to incorporate the feature of a noncommutative spacetime. This equation is much more constraining than the usual Schrödinger equation in that the spatial dimension noncommuting with time is effectively reduced to a point in low energy. We thus call the new evolution equation the spacetime bootstrap equation, the dimensional reduction called for by this evolution seems close to what is required by the holographic principle. We will discuss several examples to demonstrate this point.

Keywords: Non-Commutative Geometry, Models of Quantum Gravity.

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String theory intrinsically exhibits spacetime uncertainty, as first pointed out in [1], [2] and later examined in some nonperturbative context in [3], [4]. This seems to indicate that space and time will become noncommutative in a more fundamental formulation of string/M theory. On the other hand, quantum black hole physics also cries for noncommutative spacetime [5], [6], for a different but somewhat related reason. For a recent review on the two aspects of spacetime uncertainty, see [7]. These observations recently led many people to believe that in string theory and more generally in a theory of quantum gravity, the notion of spacetime will become approximate, and that a fundamental formulation of string/M theory should unify kinematics and dynamics of spacetime into a single framework.

A noncommutative spacetime implies that some dimensions are redundant in a quantum mechanical description, much as in quantum mechanics information is required of the wave function only for one variable in a pair of conjugate variables. This, we believe, is closely related to the holographic principle in a quantum gravity theory [8] in which a few spatial dimensions are reduced. Thus, it is very likely that in a theory exhibiting holography, such as the AdS/CFT correspondence [9], space and time are noncommutative. The first concrete proposal in string theory, to the best of our knowledge, appears in [10] in which a fuzzy AdS_2 is proposed to describe the quantum gravity effects. Indeed, a correct pattern of $1/N$ expansion as well as of nonperturbative effects in large N emerges, the latter can be interpreted as caused by the instanton processes of fragmentation of smaller AdS_2.

It is possible to construct theories with noncommutative spacetime by switching on an electric field on D-branes. Unlike in the magnetic case, no decoupling limit exists for one to obtain an effective field theory [11]. If one tries to define a field theory on a noncommutative spacetime perturbatively using Feynman rules, one runs into contradiction: The theory

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would be nonunitary perturbatively [12]. Even if such inconsistency was not there, one would still have problem to define such a theory nonperturbatively. To define a quantum amplitude between two field configurations at two given times, one would have to specify many time derivatives of fields, since the action contains infinitely many time derivatives. This in a way requires one to introduce many more degrees of freedom than those the theory initially contains.

We propose to study noncommutative spacetime in the opposite direction, namely instead of introducing more degrees of freedom we shall try to reduce the number of degrees of freedom, as we explained above that interesting physics demands such a procedure. To this end, it appears to us that it is necessary to reformulate quantum mechanics when time is regarded as an operator rather than a continuous evolution parameter. As we shall explain shortly, usual quantum mechanics for a single degree of freedom can be easily formulated on a noncommutative space, and the second quantization, namely a field theory, can also be formulated. However, one can not write down a Schrödinger equation in which noncommutativity of space and time manifests, this is simply because in the such an evolution equation time is always treated as a c-number variable. We will propose a simple generalization of the time evolution equation. This equation departs from quantum mechanics in a minimal fashion, and is much more constraining than the Schrödinger equation, thus enabling dimensional reduction. If the system under study also involves other commuting spatial dimensions, we will be led to an effective low dimensional theory defined on the commuting space, in the low energy limit. In the dimensional reduced theory, usual quantum mechanics is recovered.

1. Noncommutative space

We start with the problem of formulating quantum mechanics on a noncommutative plane. This subject has not received much attention except [13], the reason is probably that for many experts it is rather straightforward to do so. We present a discussion here for the purpose to distinguish this case from our subject of main interest: noncommutative spacetime. Our discussion is somewhat different from that in [13]. Let us start with the following operator algebra

$$[\hat{x}^i, \hat{x}^j] = i\epsilon^{ij}\theta\,, \qquad [\hat{x}^i, \hat{p}_j] = i\delta_{ij}\,, \qquad [\hat{p}_i, \hat{p}_j] = 0\,. \tag{1.1}$$

Since $\hat{x}^i$ are noncommuting, they do not have mutual eigenstates in the Hilbert space. $\hat{p}_i$ however are still commuting, they do have mutual eigenstates

$$\hat{p}_i|p\rangle = p_i|p\rangle\,, \tag{1.2}$$

or for a state $|\Psi\rangle$ expanded in the above basis $\Psi(p) = \langle p|\Psi\rangle$:

$$\hat{p}_i\Psi(p) = p_i\Psi(p)\,. \tag{1.3}$$

The multiplication on the r.h.s. is the usual one.

In the momentum representation, the position operators are realized by

$$x^i = i\partial_{p_i} - \frac{\theta}{2}\epsilon^{ij}p_j\,, \tag{1.4}$$

when acting on $\Psi(p)$. Now although it is impossible to define the eigenstates of operators $\hat{x}^i$, one can define the state $|x\rangle$ by $\langle x|p\rangle = \exp(ip_ix^i)$ so that

$$\hat{x}^i\Psi(x) = x_i\Psi(x) + \frac{i\theta}{2}\epsilon^{ij}\partial_j\Psi(x) = x_i * \Psi(x)\,, \tag{1.5}$$

where we used the standard $*$-product.

It is not hard to convince oneself that for any operator as a function of $\hat{x}^i$, if Weyl ordered, its action on the wave function $\Psi(x)$ is given by the $*$-product between itself and the wave function. If there is a potential for a single particle depending only on $\hat{x}$, $V(\hat{x})$, then one has $V(\hat{x})\Psi(x) = V(x) * \Psi(x)$. The Schrödinger equation can be written as

$$i\partial_t\Psi(t,x) = \left(-\frac{1}{2m}\partial_i^2 + V(x)*\right)\Psi(t,x)\,. \tag{1.6}$$

Apparently, there are just as many solutions to the above equation as many states in the Hilbert space, and the latter is the space of a certain class of functions of x. This means that although the plane is noncommutative, there is no dimensional reduction at all.

In the second quantization, both time and space coordinates become dummy variables, and it is the field itself acts as an operator. For a nonrelativistic particle moving on a noncommutative plane and under a potential, the hamiltonian is written as

$$H = \int d^2x\left(-\frac{1}{2m}\overline{\Psi}(x)\partial_i^2\Psi(x) + \overline{\Psi}(x) * V(x) * \Psi(x)\right). \tag{1.7}$$

If the interaction is a type of self-interaction, one replaces $V(x)$ in (1.7) by an operator such as $\overline{\Psi}(x) * \Psi(x)$, one ends up with a noncommutative field theory.

2. Noncommutative spacetime

We have seen that all is fine with a noncommutative space. The standard quantum mechanics applies rather straightforwardly. We now postulate that for some reason time becomes noncommutative with a spatial dimension x. We have an operator algebra

$$[\hat{t},\hat{x}] = i\theta, \quad [\hat{x},\hat{p}] = i\,. \tag{2.1}$$

If we set the speed of light $c = 1$, θ has a dimension of length square. Apparently, there is trouble with the usual idea about the Hilbert space. Assume that the Hilbert space is the usual representation of the Heisenberg algebra $[\hat{x},\hat{p}] = i$, then since $[\hat{x},\hat{t}+\theta\hat{p}] = 0$, one deduces

$$\hat{t} = -\theta\hat{p} + f(\hat{x})\,, \tag{2.2}$$

that is, time is no longer an independent dummy variable, and is proportional to momentum. This in itself is correct for many physics phenomena, but impedes generalization of the Schrödinger equation. For now the standard wave function $\Psi(t,x)$ would become a function of the form $\Psi(p,x)$. This does not make sense for Ψ as a wave function. Only the Wigner distribution depends both on x and p. If on the other hand one insists on treating the wave function as a function of only one variable, then in the standard Schrödinger equation the commutator $[\hat{t},\hat{x}]=i\theta$ plays no role at all.

Thus it seems necessary to go beyond quantum mechanics, as first emphasized in the conclusion section of [7]. We will still postulate that it makes sense to talk about the joint function $\Psi(t,x)$, but one has to generalize the Schrödinger equation. The right version of the time evolution equation to be generalized seems to be the integrated form

$$\Psi(t,x)=e^{-itH(\hat{p},\hat{x})}\Psi(x)\,. \tag{2.3}$$

Now for *any* initial wave function, there is no problem in using the above equation to evolve it to time t. Due to the simple fact that

$$e^{-it'H}e^{-itH}=e^{-i(t+t')H}\,, \tag{2.4}$$

one has the composition law

$$e^{-it'H}\Psi(t,x)=\Psi(t+t',x)\,, \tag{2.5}$$

as demanded by the requirement that the fundamental evolution equation must be time-translationally invariant.

Now when $\hat{t}$ is noncommuting with $\hat{x}$, one might be motivated to generalize (2.3) to the form

$$\Psi(t,x)=e^{-itH(\hat{p},x)}*\Psi(x)\,, \tag{2.6}$$

where $\Psi(x)$ is the wave function at $t=0$. The above $*$-product is a generalized one defined by

$$f(t',x)*g(t,y)=e^{\frac{i\theta}{2}(\partial_{t'}\partial_y-\partial_t\partial_x)}f(t',x)g(t,y)\,. \tag{2.7}$$

Such a generalized $*$-product for functions at different points have been extensively used recently in constructing gauge invariant observables in a noncommutative Yang-Mills theory. Although in (2.6) the two times are different, the space points in two functions are the same. Also we implicitly assume in (2.6) that $\hat{p}$ is identified with $-i\partial_x$, and in the $*$-product it is treated as a dummy variable.

Evolution equation (2.6) differs from the quantum mechanical one (2.3) in that we need to specify all the time derivatives of $\Psi(x)$ at the initial time $t=0$. This at the first sight requires introducing more degrees of freedom. We shall see shortly that the opposite is true. The evolution form is not guaranteed to be time-translationally invariant, since

$$e^{-it'H(\hat{p},x)}*e^{-itH(\hat{p},x)}\neq e^{-i(t+t')H(\hat{p},x)}\,. \tag{2.8}$$

Thus, instead of working with (2.6) we propose to work with the following equation

$$\Psi(t'+t,x)=e^{-it'H(\hat{p},x)}*\Psi(t,x)\,. \tag{2.9}$$

The above equation is required to be valid for arbitrary t and t', therefore by definition the composition law is satisfied. (2.9) differs from (2.6) in that on the r.h.s. of (2.6) Ψ and its time derivatives can be arbitray at $t = 0$, while (2.9) is better viewed as a contraining equation rather than an evolution equation, since one can not arbitrarily specify Ψ and its time derivatives at t, their form must be consistent with the function on the l.h.s. We shall see soon that indeed this equation will drastically reduce the number of solutions. If one views (2.9) both as an evolution equation as well as a constraining equation, it is better called the bootstrap equation, in the sense that time and space bootstrap themselves.

By the definition of the generalized $*$-prouduct, we have the following algebra

$$[t, x] = [t', x] = i\theta \qquad [t, t'] = 0\,, \tag{2.10}$$

where we omitted hat for all variables. This algebra is certainly consistent with the one we started with in (2.1). We now proceed to examine the implications of our main equation (2.9).

3. $H(p, x) = H(p)$

The simplest hamiltonian to work with is a hamiltonian depending only on momentum. In this case, (2.9) has the form

$$\Psi(t + t', x) = e^{-it'H(p)}\Psi(t, x) + \sum_{n\geq 1} \frac{1}{n!}\left(\frac{\theta H}{2}\right)^n e^{-it'H}\partial_x^n \Psi(t, x)\,. \tag{3.1}$$

At $t' = 0$, we deduce

$$\sum_{n\geq 1} \frac{1}{n!}\left(\frac{\theta H}{2}\right)^n \partial_x^n \Psi(t, x) = 0\,. \tag{3.2}$$

Once this constraining equation is satisfied, the sum in (3.1) automatically vanishes, and we have

$$\Psi(t + t', x) = e^{-it'H(p)}\Psi(t, x)\,, \tag{3.3}$$

the standard quantum mechanical evolution equation. Both the constraining equation (3.2) and (3.3) are linear, thus an arbitrary solution can be decomposed into a complete basis of solutions. Any solution in this basis satisfies the composition law, since (3.3) is the usual quantum mechanical equation.

Without loss of any information, we will set $t = 0$ in (3.2) and proceed to solve it. Denote the wave function at $t = 0$ by $\Psi(x)$. Using Fourier transformation

$$\Psi(x) = \int dp\tilde{\Psi}(p)e^{ipx}\,, \tag{3.4}$$

we find

$$\int \left(e^{i\frac{\theta}{2}pH(p)} - 1\right)\tilde{\Psi}(p)e^{ipx}dp = 0\,. \tag{3.5}$$

So the constraining equation boils down to the simple equation

$$\left(e^{i\frac{\theta}{2}pH(p)} - 1\right)\tilde{\Psi}(p) = 0\,. \tag{3.6}$$

For $H(p)$ non-singular at $p = 0$, there is a obvious solution $\Psi(p) = \delta(p)$. If $H(p)$ also vanishes at $p = 0$, there are more solutions. For instance for a "nonrelativistic" particle $H(p) = p^2/(2m)$, there are three solutions localized at $p = 0$:

$$\Psi(p) = a\delta(p) + b\delta'(p) + c\delta''(p)\,. \tag{3.7}$$

In the position space, we have

$$\Psi(x) = a + bx + cx^2\,. \tag{3.8}$$

After applying the evolution equation (3.3), we get

$$\Psi(x) = a + bx + c\left(x^2 + \frac{it}{m}\right). \tag{3.9}$$

It is easy to check that indeed this family of solutions satisfies the bootstrap equation (2.9). The energy of these solutions is zero. We conclude that in the low energy sector, the number of solutions is finite. If $H(p) \sim p^m$, there are $m+1$ solutions in the zero-momentum sector.

For nonvanishing p, the factor in the parenthesis in (3.6) vanishes if

$$p_n H(p_n) = \frac{4\pi n}{\theta}\,, \tag{3.10}$$

so there is a solution $\Psi(p) \sim \delta(p - p_n)$. The momentum is quantized, although we start with a noncompact dimension x! The simplest case is when H is a constant $\mathcal{E}$,

$$p_n = \frac{4\pi n}{\theta\mathcal{E}}\,. \tag{3.11}$$

This formula indicates that the particle effectively moves on a circle of radius

$$R = \frac{\theta\mathcal{E}}{4\pi}\,. \tag{3.12}$$

The smaller the parameter θ is, the smaller the circle, the larger the momentum, while the energy is always a constant. Since in string theory θ is related either to the string scale or the Planck scale, the nonzero momentum is actually quite high, one might say that the low momentum sector (p=0) is one dimensional. Even when one takes all nozero momentum (3.11) into account, the original quantum mechanical Hilbert space is reduced to one in which momentum is discrete. We see that indeed we achieved what we hoped: There is a drastic reduction of degrees of freedom.

The next interesting case is when $H(p) = |p|$. We find

$$p_n^2 = \frac{4\pi n}{\theta}\,. \tag{3.13}$$

This quantization condition is similar to the string spectrum.[1] In this case, the energy is no longer constant, but proportional to $\sqrt{n/\theta}$. Again, for small θ, all these are in the high energy sector. There are two states in the low energy sector.

[1] I am grateful to P. M. Ho for suggesting this analogue.

If we have a "nonrelativistic" particle, we have

$$p_n^3 = \frac{8\pi mn}{\theta} \,. \tag{3.14}$$

In this case there is no direct analogue in familar models.

We notice that as long as $\theta \neq 0$, the nonvanishing momentum is always quantized. This phenomenon has no smooth limit at $\theta = 0$, where the constraining equation (3.2) is always satisfied for any p.

Suppose in addition to x, there are more spatial dimensions and these are always commuting with time. Assume $H = H(p, q)$, q is the momentum in the commuting directions. The quantization condition (3.10) is still valid. In this case p_n depends continuously on q, in general we have $p_n = p(n, q)$. q for a given n can vary in a range continuously. For example, if

$$H = \sqrt{p^2 + q^2} \,, \tag{3.15}$$

then

$$p_n^2 = \frac{1}{2}\left(\left(q^4 + 4\left(\frac{4\pi n}{\theta}\right)^2\right)^{1/2} - q^2\right). \tag{3.16}$$

For small q, namely $q^2 \ll 1/\theta$

$$p_n^2 \sim \frac{4\pi|n|}{\theta} \,. \tag{3.17}$$

The energy is also of this order. For large q, that is $q^2 \gg 1/\theta$,

$$p_n \sim \frac{4\pi|n|}{|q|\theta} \,. \tag{3.18}$$

The energy is approximately $|q|$. In both cases, the energy is no smaller than $1/\sqrt{\theta}$.

We are more interested in the low energy sector, namely when $p = 0$. For the above relativistic model, there is always only one state if $q \neq 0$. In this case $E = |q|$, the theory is completely reduced to a zero mass relativistic particle in the commuting space, and moreover the usual quantum mechanics is valid for this dimensionally reduced theory.

One can proceed to discuss different hamiltonians when q is involved. Suffices it to say that in addition to the low energy sector which is finite dimensional in terms of x-dimension, all other solutions are highly energetic if θ is very small. In all these theories, the dimensional reduction occurs in the p-space.

4. $H = H(x)$

Next we consider the case when the hamiltonian is a function of only x. In order to solve the bootstrap equation (2.9) completely, we need the following formula for the $*$-product,

$$\begin{aligned} f(t', x) * g(t, x) &= \frac{1}{(\theta\pi)^2}\int \prod_i dt_i dx_i f(t_1, x_1) g(t_2, x_2) \times \\ &\quad \times \exp\left(\frac{2i}{\theta}\left[(t' - t_1)(x - x_2) - (t - t_2)(x - x_1)\right]\right). \end{aligned} \tag{4.1}$$

For $f(t_1, x_1) = \exp(-it_1 H(x_1))$ and $f(t_2, x_2) = \Psi(t_2, x_2)$, one may integrate out t_1 and then x_2 first to find the bootstrap equation

$$\Psi(t+t', x) = \frac{1}{\theta\pi}\int dt_2 dx_1 \Psi\left(t_2, x + \frac{\theta}{2}H(x_1)\right)\exp\left(-it' H(x_1) - \frac{2i}{\theta}(t - t_2)(x - x_1)\right). \tag{4.2}$$

Use the definition

$$\tilde{\Psi}(E, x) = \frac{1}{2\pi}\int dt e^{iEt}\Psi(t, x) \tag{4.3}$$

to integrate out t_2

$$\Psi(t+t', x) = \frac{2}{\theta}\int dx_1 \tilde{\Psi}\left(\frac{2}{\theta}(x - x_1), x + \frac{\theta}{2}H(x_1)\right)\exp\left(-it' H(x_1) - \frac{2i}{\theta}t(x - x_1)\right). \tag{4.4}$$

Fourier transforming both sides of the above equation with respect to t results in

$$e^{-it'E}\tilde{\Psi}(E, x) = e^{-it' H(x - \frac{\theta}{2}E)}\tilde{\Psi}\left(E, x + \frac{\theta}{2}H\left(x - \frac{\theta}{2}E\right)\right). \tag{4.5}$$

The functional relation (4.5) is easy to solve. If H is a nontrivial function of x, the t' dependence of the two sides of (4.5) do not agree unless $\tilde{\Psi}(E, x)$ concentrates at the point when $H(x - \frac{\theta}{2}E) = E$, that is

$$\tilde{\Psi}(E, x) \sim \delta\left(H\left(x - \frac{\theta}{2}E\right) - E\right). \tag{4.6}$$

Compare the x dependence of the two sides of (4.5) we find that whenever

$$H\left(x - \frac{\theta}{2}E\right) = E \tag{4.7}$$

x and E must also satisfy

$$H\left(x + \frac{\theta}{2}H\left(x - \frac{\theta}{2}E\right) - \frac{\theta}{2}E\right) = E. \tag{4.8}$$

Substituting $H(x - \frac{\theta}{2}E)$ in the above relation by E, we have

$$H(x) = E.$$

We conclude that the following two equations must be satisfied simultaneously

$$H\left(x - \frac{\theta}{2}E\right) = E, \qquad H(x) = E. \tag{4.9}$$

For a monotonic function $H(x)$, the two equations in (4.9) can not possibly be satisfied at the same time unless $E = 0$. Thus, we have

$$\tilde{\Psi}(E, x) = f(x)\delta(H(x))\delta(E). \tag{4.10}$$

To determine the form of $f(x)$, substituting the above ansatz back to (4.5) we find

$$f(x)\delta(H(x)) = f\left(x + \frac{\theta}{2}H(x)\right)\delta\left(H\left(x + \frac{\theta}{2}H(x)\right)\right). \tag{4.11}$$

This equation does not always have a solution. For instance if $H = \omega^2 x$, there is no solution. It is easy to convince oneself that any zero of $H(x) = 0$ must be of higher order. There is a solution when $H = \omega^2 x^3$, in this case $f(x) \sim x$. In other word,

$$\tilde{\Psi}(E, x) = \delta(E)\delta(x)\,. \tag{4.12}$$

The solution is completely localized at $x = 0$ and the energy vanishes there. This reduction of degrees of freedom is even more complete than in cases when H is a function of momentum p.

If $H(x)$ is not a monotonic function of x, there can be more than one solutions. Consider $H(x) = \omega^2 x^2$. The two equations of (4.9) can be solved if $E = 0$ and $\tilde{\Psi}$ is localized at $x = 0$. These equations can also be solved at the same time if

$$E = \left(\frac{4}{\theta\omega}\right)^2, \tag{4.13}$$

and Ψ is localized at

$$x = \frac{4}{\theta\omega^2}\,. \tag{4.14}$$

However this solution is not good, as for the value (4.13), there are more zeros of (4.8) than those of (4.7). In other words, to have nonvanishing energy in a soluton, we need a complicated function $H(x)$. As a more interesting example, take

$$H(x) = \mathcal{E}\sin(\omega x)\,. \tag{4.15}$$

This function has a period $2\pi/\omega$, so a class of solutions is

$$E_n = \frac{4\pi n}{\theta\omega}, \qquad \sin(\omega x_n) = \frac{4\pi n}{\theta\omega\mathcal{E}}\,. \tag{4.16}$$

When $\theta\omega\mathcal{E} \gg 1$, there are many such localized solutions.

Now if there are commuting spatial dimensions, and H depends also on either the position in this space or associated momentum, or both, there can be solutions to the two equations of (4.9). In this case, both E and the localization x will depend on these extra parameters. If H is always a monotonic function of x, no interesting solution will be obtained, since E always vanishes thus is independent of the parameters in the commuting space. If H is not a monotonic function, one can have some fun here. For instance, replace ω^2 in the hamiltonian $H = \omega^2 x^2$ by $c\theta^{-2}|q|^{-1}$, where q is the momentum along the commuting space and c is a dimensionless parameter, we have

$$E = \frac{16}{c}|q|\,, \qquad x = \frac{4}{c}\theta|q|\,. \tag{4.17}$$

The first identity in the above is a relativistic dispersion relation, while the second relation looks like the UV/IR relation in the AdS/CFT correspondence. Of course we can not possibly hope to get a realistic model of AdS/CFT correspondence in this fashion any time soon. The relations in (4.17), though encouraging, can not be taken literally as we remarked before.

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5. The general case

We have seen that when H is a function of p only, the wave functions are localized in the p space; when H is a function of x only, the wave functions are localized in the x space, and the number of possible solutions is more reduced in this case. We expect that more interesting phenomenon will happen if H is a function of both p and x. This problem can be hardly solved generally, so we will be content with a remark.

Similar to (4.2), for $H(p, x)$, the bootstrap equation (2.9) can be written in an integral form

$$\Psi(t+t', x) = \int dt_2 dx_2 G(t, t', x, t_2, x_2)\Psi(t_2, x_2)\,. \tag{5.1}$$

This in general is a difficult integral equation to solve. For example, for H to be the hamiltonian of a harmonic oscillator, we do not know how to solve this equation at present. The kernel G in no way depends on t and t' in the combination $t+t'$, so the composition law in time is a rather strong constraint on the possible wave functions. We get a projection equation at $t' = 0$, this requires that the kernel $G(t, 0, x, t_2, x_2)$ is a projection operator, and in general it has only a finite dimensional invariant subspace of the space of all functions. So the dimensional reduction is a general mechanism.

6. Conclusion and discussions

First of all, we want to emphasize that there is no issue of violating unitarity in the approach we suggest here. In our view, as we explained in the beginning of this paper, the usual action principle for a field theory on a noncommutative spacetime is an ill-defined notion. One may try to get rid of the conceptual issues as well as the technical ones by introducing more degrees of freedom, or going into higher dimensions, as advocated in [14]. This attempt runs in the opposite direction to our philosophy.

The quick investigation of the idea about going beyond quantum mechanics for a noncommutative spacetime in this paper leaves many problems unanswered. On the technical side, one would like to solve the bootstrap equation for more hamiltonians to gain experience and insight about how dimensional reduction occurrs. The bootstrap equation proposed here does not have a differential form, one that would be close to the original Schrödinger equation. The best one can hope for is to formulate a path integral form, thus spelling out this procedure in an action principle.

It is also interesting to see whether the simplest form of the bootstrap equation (2.9) has other variants. An important point is that not only one wants to incorporate the noncommutative spacetime, one also wants to reduce to quantum mechanics after the dimensional reduction is achieved. It therefore appears that not many such variants exist.

To make contact with field theories and string theory, ultimately one needs to study second quantization. Now wave functions are promoted to operators, these operators again should satisfy the bootstrap equation when on-shell. Now a second quantized hamiltonian is to be constructed in such a way that the bootstrap equation is the equation of motion resulting from it. Or is there a second quantized hamiltonian at all?

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To study the AdS/CFT correspondence, in particular to understand how holography really works, there is a long way to go. Apparently in the usual formulation of AdS/CFT, the two sides have different dynamical variables and hamiltonians, although there is a dictionary between them. As suggested in [10], the noncommutative spacetime is a notion for the bulk theory, so when applying the present idea to that context, the resulting boundary theory may well be different from the known CFT. It might be that there is a set of bulk variables different from those known in the bulk theory (more fundamental variables in some sense?), such that when dimensionally reduced, one gets the CFT.

It has been advocated by some people for some time that in a fundamental formulation of the bulk variables in string/M theory, there is a large gauge symmetry. Once the symmetry is fixed, one is led to the boundary theory just as what happens in Chern-Simons theory with a boundary. This suggestion also demands to construct new bulk variables, just as in our proposal. We however view our proposal as a more attractive one for several reasons:

1. Noncommutative spacetime by now is commonly believed to be a general feature of string/M theory, apparently it must be linked to holography. Our proposal shows that indeed this is the case.

2. In the bootstrap equation, quantum mechanics plays an essentially role. Spacetime uncertainty is a dynamical phenomenon, in the very beginning we build this in the fundamental equations. That AdS/CFT works precisely because on each side quantum mechanics is taken into account. This goes without saying on the CFT side. On the AdS side, for example, the giant graviton phenomenon is a quantum mechanical, nonperturbative one.

3. In our scheme we are beginning to see how spacetime and quantum mechanics are intimately tied up. Also, what is more exciting than going beyond quantum mechanics?

4. Finally, we have presented a concrete and very simple scheme, it would be wasteful if we do not explore it in depth.

Finally, it is important to point out that while in the present proposal the dimensional reduction is achieved, the cost one has to pay is the difficulty in recovering usual quantum mechanics in a proper limit. Our conjecture on remedy of this problem is that the usual quantum/classical dynamics in the bulk in a holographic theory can be recovered only in the large N limit in the boundary theory, so it may be true also in our proposal that in order to recover the standard quantum mechanics in the bulk theory we need to generalize our formalism to include many degrees of freedom. Then, for instance, a graviton as a highly composite object (it corresponds to a single trace operator in the AdS/CFT correspondence) may be found to obey the usual quantum mechanics in the large N limit.

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